



Introduction to Embodied AI

Lecture 2 - Robotics I

Prof. He Wang



Course Organization: Robotics Section

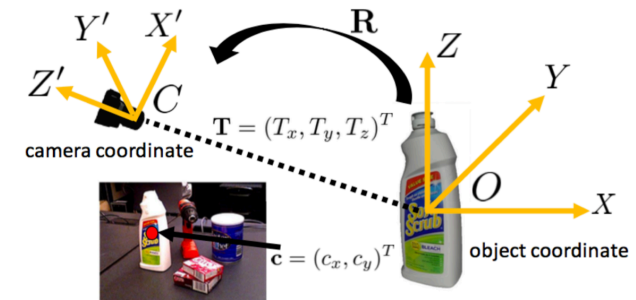
Lecture	02/24/2025 15:10 Monday	Basics of Robotics I	Kinematics of Rigid Bodies and Articulated Objects
Lecture	03/03/2025 15:10 Monday	Basics of Robotics II	Motion Planning
Lecture	03/10/2025 15:10 Monday	Basics of Robotics III	Control Algorithms (PD, PID)

(Tentative schedule)

For the complete and up-to-date schedule, see https://pku-epic.github.io/Intro2EAI_2025/schedule/.

The Basics of Robotics

- Kinematics (today's topic)
 - Rigid objects
 - Articulated objects
- Dynamics



A Typical Syllabus of Introduction to Robotics

- Robotics is a branch of engineering that involves the conception, design, manufacture and operation of robots.

Introduction to Robotics

CS223A

Stanford School of Engineering

Topics include

- Spatial descriptions
- Kinematics
- Jacobians
- Statics
- Dynamics
- Motion planning and control
- Trajectory generation

Kinematics vs. Dynamics

- Kinematics: describing the motion of bodies (position, linear/angular velocity, linear/angular acceleration, etc.)
- Kinematics **does not consider** how to achieve motion via force
- On the contrary, dynamics models all the way from force and torque to the motion

Rigid Transformation

Slides mainly borrowed from UCSD CS291-E00

Link and Joint

Link:

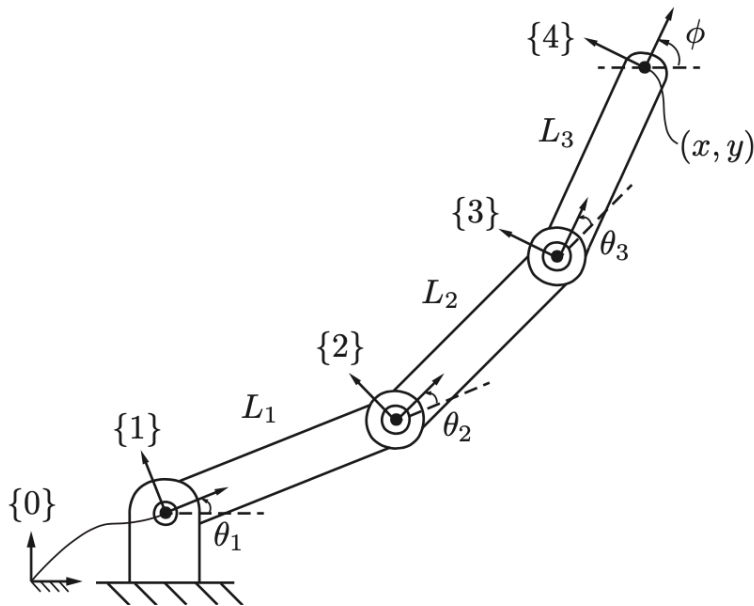
- **Links** are the rigid-body connected in sequence

Joint:

- **Joints** are the connectors between links.

DoF:

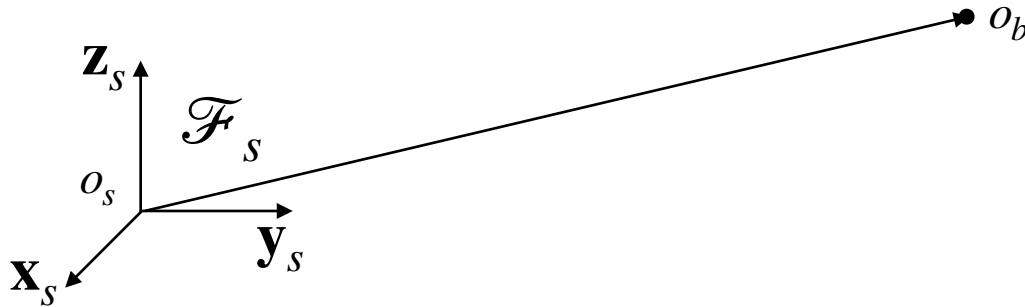
- **The degree of freedom (DoF)** of a mechanical system is the number of independent parameters that define its configuration.



Agenda

- Rigid Transformation
- $(R_{s \rightarrow b}, \mathbf{t}_{s \rightarrow b})$ for Coordinate Transformation
- $(R_{s \rightarrow b}, \mathbf{t}_{s \rightarrow b})$ as a Linear Transformation

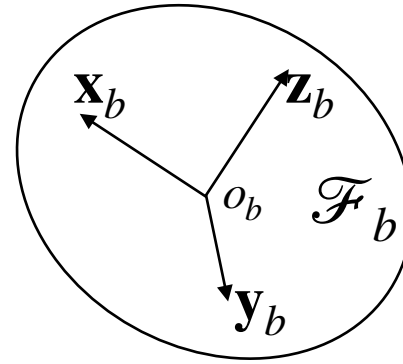
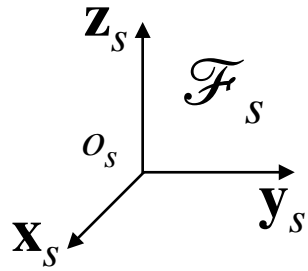
Notation Convention



- An observer **records** the position of any point in the space **using a frame** $\mathcal{F}_s$
- We use ordinary letters to denote points (e.g., p), and bold letters to denote **vectors** (e.g., $\mathbf{v}$)
- When **writing equations**, we add a superscript symbols to denote the recording frame, e.g.,

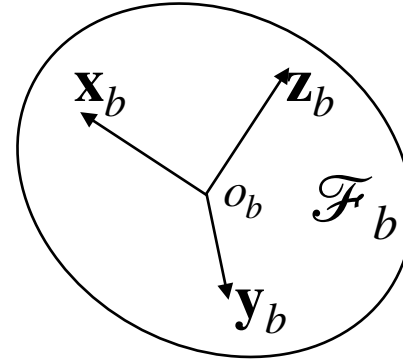
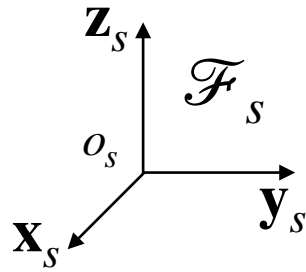
$$o_b^s = o_s^s + \mathbf{t}_{s \rightarrow b}^s$$

Rigid Transformations



- There is a rigid object, to which we bind a frame $\mathcal{F}_b$ (body frame) tightly, so that $\mathcal{F}_b$ moves along with the object

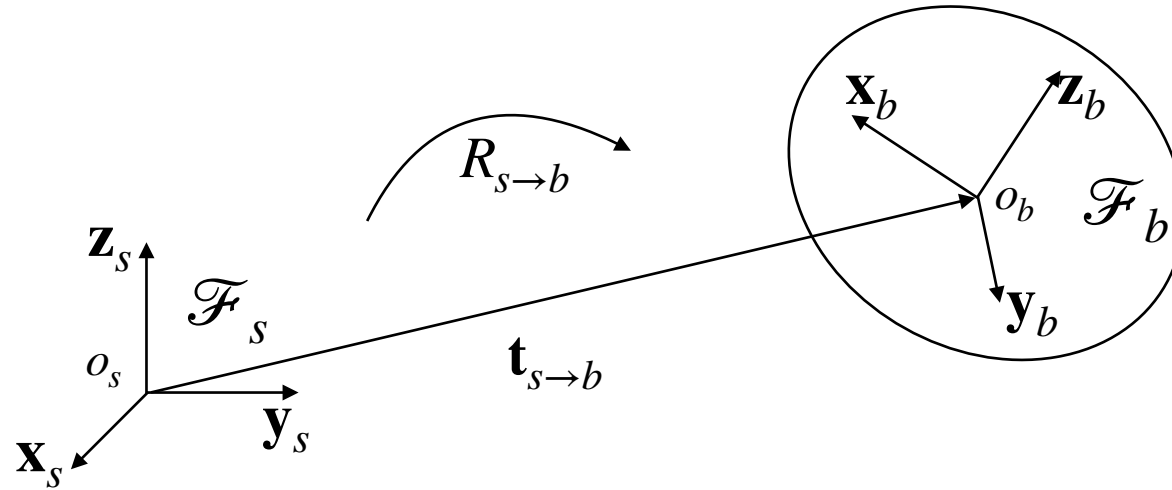
Rigid Transformations



- When talking about the pose of the *rigid* object, we ask:

How to **transform** $\mathcal{F}_s$ so that it overlaps with $\mathcal{F}_b$?

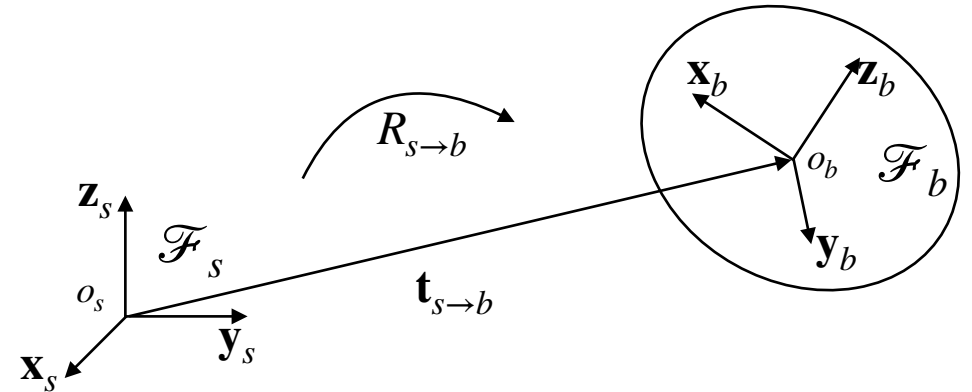
Rigid Transformations



- We first rotate by $R_{s \rightarrow b}$ to align $\{\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i\}$ ($i = s$ or b)
- And then translate $\mathcal{F}_s$ by $\mathbf{t}_{s \rightarrow b}$ to align o_s and o_b

Rigid Transformations

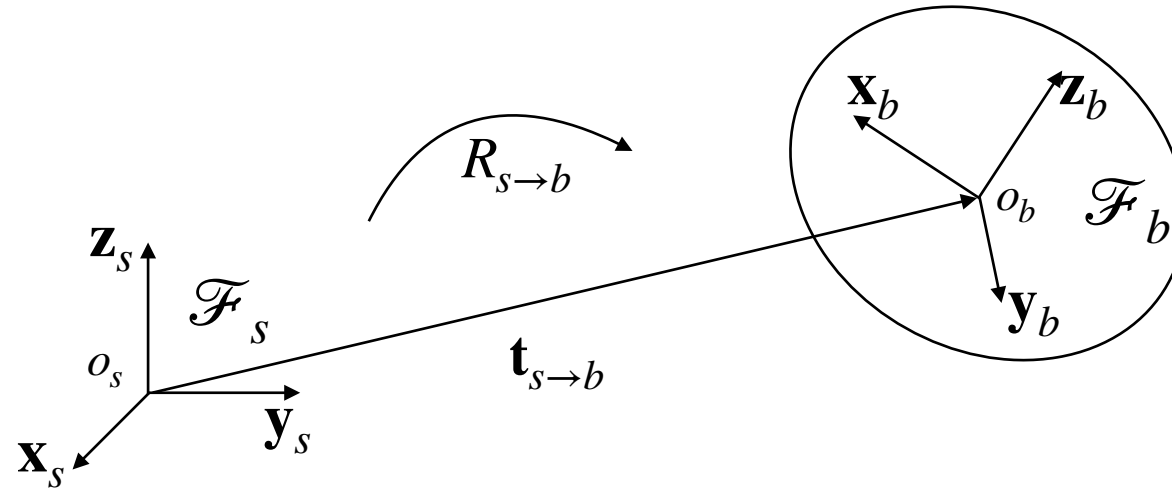
- Formally,
 - $o_b^s = o_s^s + \mathbf{t}_{s \rightarrow b}^s$
 - $[\mathbf{x}_b^s, \mathbf{y}_b^s, \mathbf{z}_b^s] = R_{s \rightarrow b}^s [\mathbf{x}_s^s, \mathbf{y}_s^s, \mathbf{z}_s^s]$
- Since the observer records everything using $\mathcal{F}_s$,
 - $o_s^s = 0$
 - $[\mathbf{x}_s^s, \mathbf{y}_s^s, \mathbf{z}_s^s] = I_{3 \times 3}$
- Therefore,
 - $\mathbf{t}_{s \rightarrow b}^s = o_b^s$
 - $R_{s \rightarrow b}^s = [\mathbf{x}_b^s, \mathbf{y}_b^s, \mathbf{z}_b^s] \in \mathbb{R}^{3 \times 3}$



Rigid Transformations

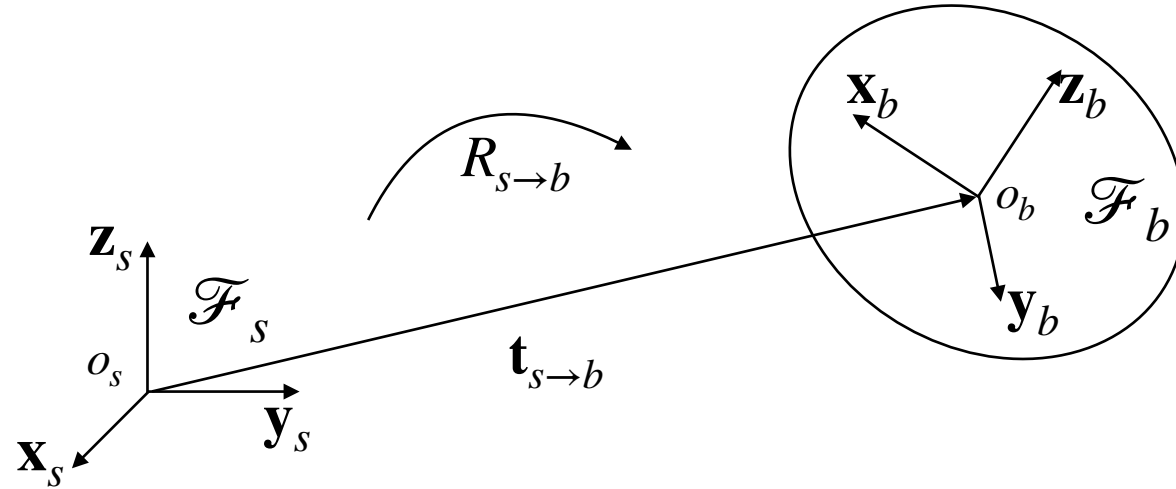
$(R_{s \rightarrow b}, \mathbf{t}_{s \rightarrow b})$ for **Coordinate Transformation**

Use Coordinate Transformation to Relate Coordinates in Frames



- Assume a second observer that records coordinates by $\mathcal{F}_b$
- Assume a point p on the body. Since $\mathcal{F}_b$ moves along the body, its coordinate recorded in $\mathcal{F}_b$, denoted as p^b , should **never change**.

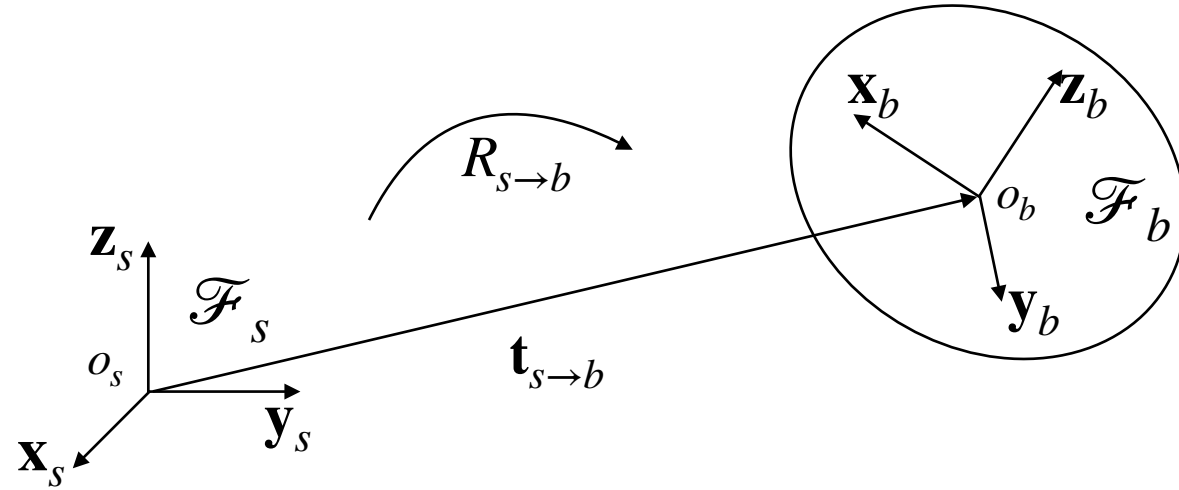
$(R_{s \rightarrow b}, \mathbf{t}_{s \rightarrow b})$ for Coordinate Transformation



- Imagine a process: $\mathcal{F}_b$ moves from $\mathcal{F}_s$ to the current location. This is how we define $(R_{s \rightarrow b}^s, \mathbf{t}_{s \rightarrow b}^s)$.
- Since p moves along $\mathcal{F}_b$, it is moved from the **initial position**, $p^s = p^b$, to the current location:

$$p^s = R_{s \rightarrow b}^s p^b + \mathbf{t}_{s \rightarrow b}^s$$

$(R_{s \rightarrow b}, \mathbf{t}_{s \rightarrow b})$ for Coordinate Transformation



- $(R_{s \rightarrow b}, \mathbf{t}_{s \rightarrow b})$ transforms any **point** in the *whole space* by the following equation:

$$\mathbf{x}'^s = R_{s \rightarrow b}^s \mathbf{x}^s + \mathbf{t}_{s \rightarrow b}^s$$

Linear Transformation

- A linear transformation between two vector spaces V and W is a map $T : V \rightarrow W$ such that the following hold:
- $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ for $\forall \mathbf{v}_1, \mathbf{v}_2 \in V$
- $T(\alpha \mathbf{v}) = \alpha T(\mathbf{v})$
- We know that we can use matrix to describe such linear transformation and use matrix multiplication to perform these linear operations.

$(R_{s \rightarrow b}, \mathbf{t}_{s \rightarrow b})$ is not Linear

- $p_1^s = R_{s \rightarrow b}^s p_1^b + \mathbf{t}_{s \rightarrow b}^s$
- $p_2^s = R_{s \rightarrow b}^s p_2^b + \mathbf{t}_{s \rightarrow b}^s$
- $p_1^s + p_2^s \neq R_{s \rightarrow b}^s (p_1^b + p_2^b) + \mathbf{t}_{s \rightarrow b}^s$ when $\mathbf{t}_{s \rightarrow b}^s \neq \mathbf{0}$
- Similarly, $\alpha p_1^s \neq R_{s \rightarrow b}^s (\alpha p_1^b) + \mathbf{t}_{s \rightarrow b}^s$ when $\mathbf{t}_{s \rightarrow b}^s \neq \mathbf{0}$
- Can we represent this transformation as a linear operation?

Homogenous Coordinates

- Homogeneous coordinate for 3D Space:

$$\tilde{x} := \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathbb{R}^4$$

- Homogeneous transformation matrix:

$$T_{s \rightarrow b}^s = \begin{bmatrix} R_{s \rightarrow b}^s & \mathbf{t}_{s \rightarrow b}^s \\ 0 & 1 \end{bmatrix}$$

- Coordinate transformation under **linear** form:

$$\tilde{x}^s = T_{s \rightarrow b}^s \tilde{x}^b$$

- Ignore $\sim$ for simplicity in the future.

Homogenous Coordinates

- The coordinate transformation works for any choice of $\mathcal{F}_s$ and $\mathcal{F}_b$
- As a general rule, we have:

$$x^1 = T_{1 \rightarrow 2}^1 x^2$$

Some Rules of Homogenous Coordinate Transformation

By $x^1 = T_{1 \rightarrow 2}^1 x^2$, we have $x^2 = T_{2 \rightarrow 1}^2 x^1$ and $x^3 = T_{3 \rightarrow 2}^3 x^2$.

Therefore, $x^3 = T_{3 \rightarrow 2}^3 T_{2 \rightarrow 1}^2 x^1$. But $x^3 = T_{3 \rightarrow 1}^3 x^1$

- Composition rule: $T_{3 \rightarrow 1}^3 = T_{3 \rightarrow 2}^3 T_{2 \rightarrow 1}^2$

By $x^1 = T_{1 \rightarrow 2}^1 x^2$, we have $x^2 = (T_{1 \rightarrow 2}^1)^{-1} x^1$

- Change of observer's frame: $T_{2 \rightarrow 1}^2 = (T_{1 \rightarrow 2}^1)^{-1}$

Multi-Link Rigid-Body Geometry

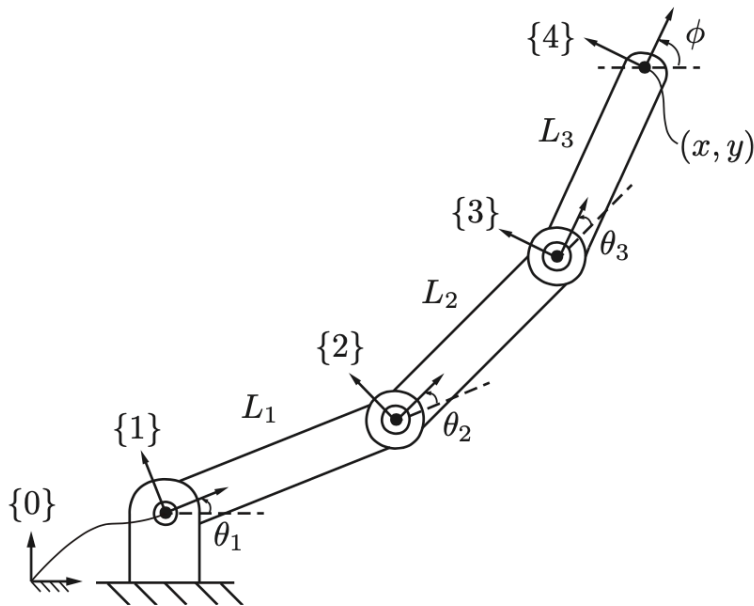
Link and Joint

Link:

- **Links** are the rigid-body connected in sequence

Joint:

- **Joints** are the connectors between links.



Joints

Joint Limits:

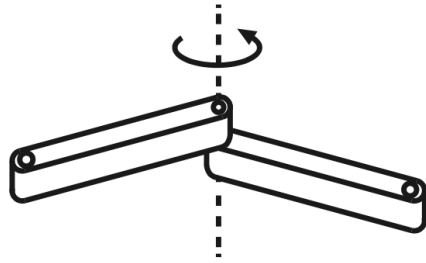
- the lower and upper limits (in *radians* for revolute joints, in *metres* for prismatic joints).

DoF:

- **The degree of freedom (DoF)** of a mechanical system is the number of independent parameters that define its configuration.

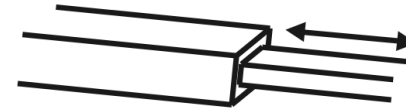
Two Common Joint Types

- Revolute/Hinge/Rotational joint



Revolute (R)
1 DoF

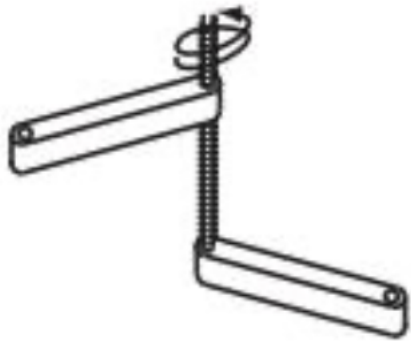
- Prismatic/Translational joint



Prismatic (P)
1 DoF

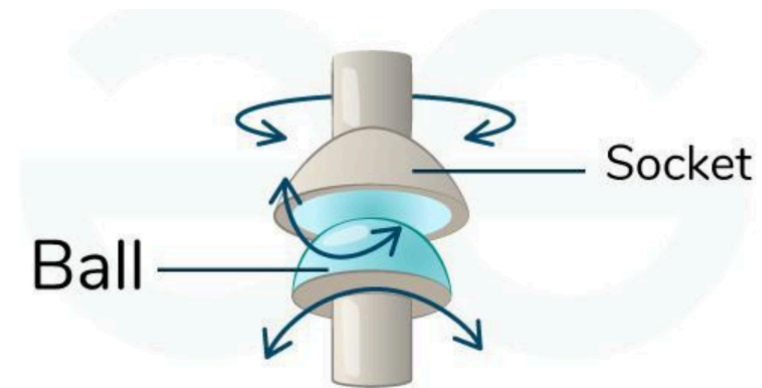
Other Joint Types

- Helical joint



Helical(H)
1 DoF

- Spherical/Ball and Socket joint



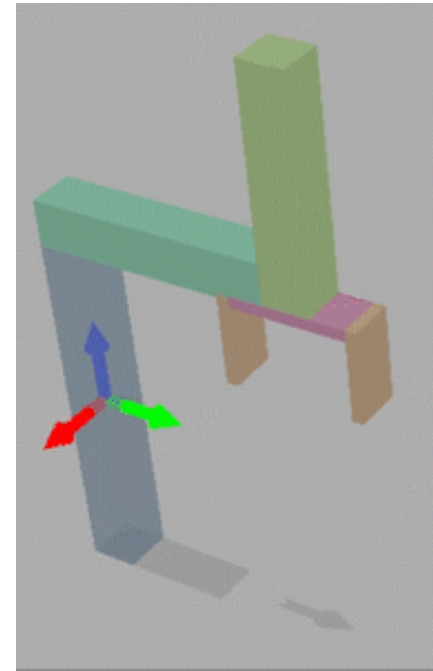
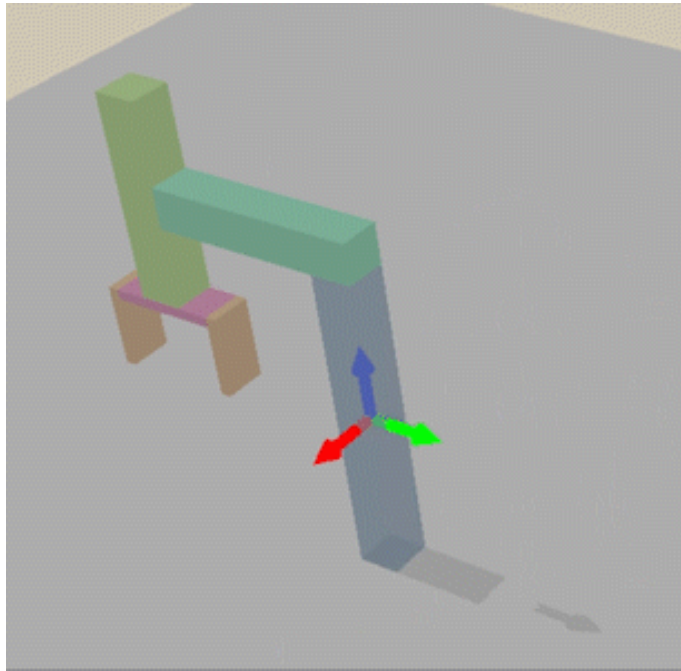
Spherical (S, a ball fits into a
spherical socket.)
3 DoF

Base Link

- Base link / root link:
 - The 0-th link of the robot
 - Regarded as the “fixed” reference
 - The spatial frame $\mathcal{F}_s$ is attached to it

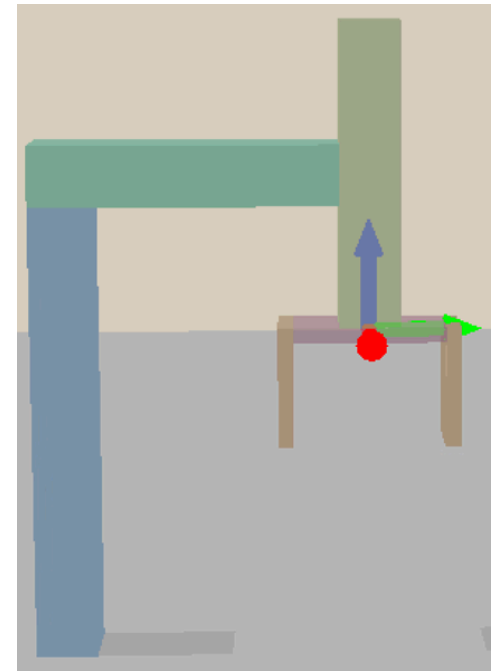
Examples

A simple 2 DoF robot arm

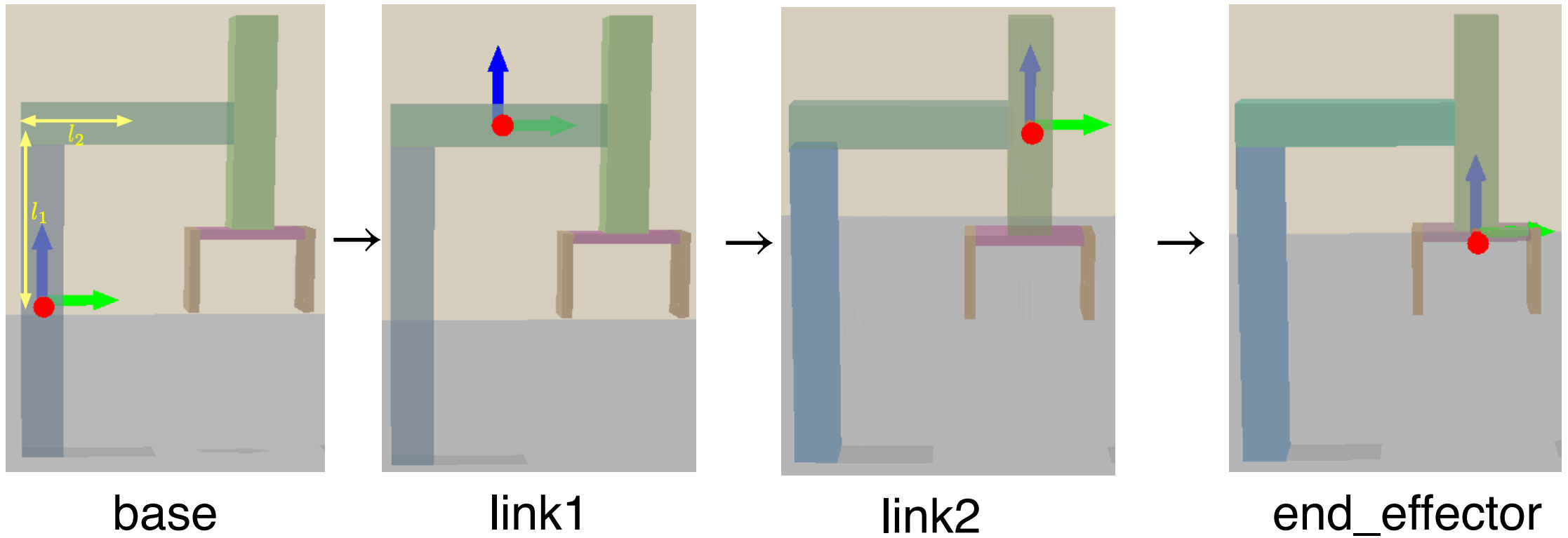


Base Link and End-Effector Link

- Base link / root link:
 - The 0-th link of the robot
 - Regarded as the “fixed” reference
 - The spatial frame $\mathcal{F}_s$ is attached to it
- End-effector link
 - The last link
 - e.g., the gripper
 - A frame $\mathcal{F}_e$ is attached to it

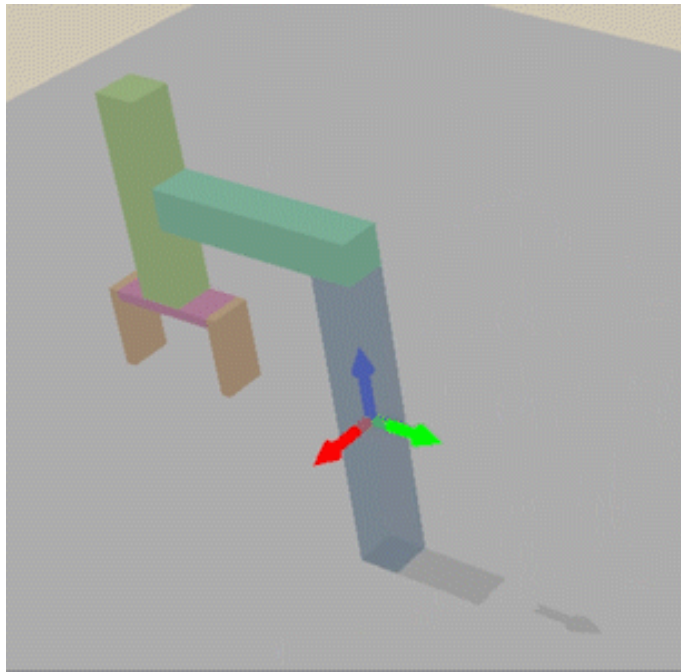


Examples

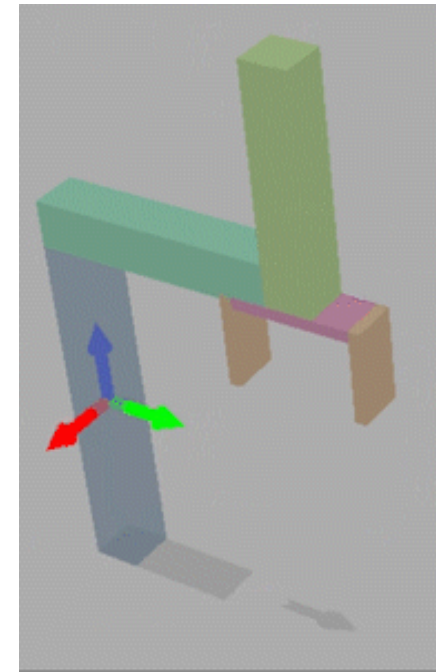


Examples

A simple 2 DoF robot arm



revolute (θ_1)



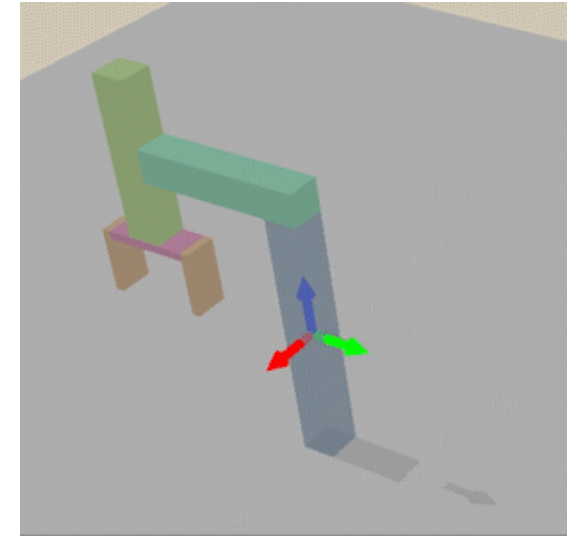
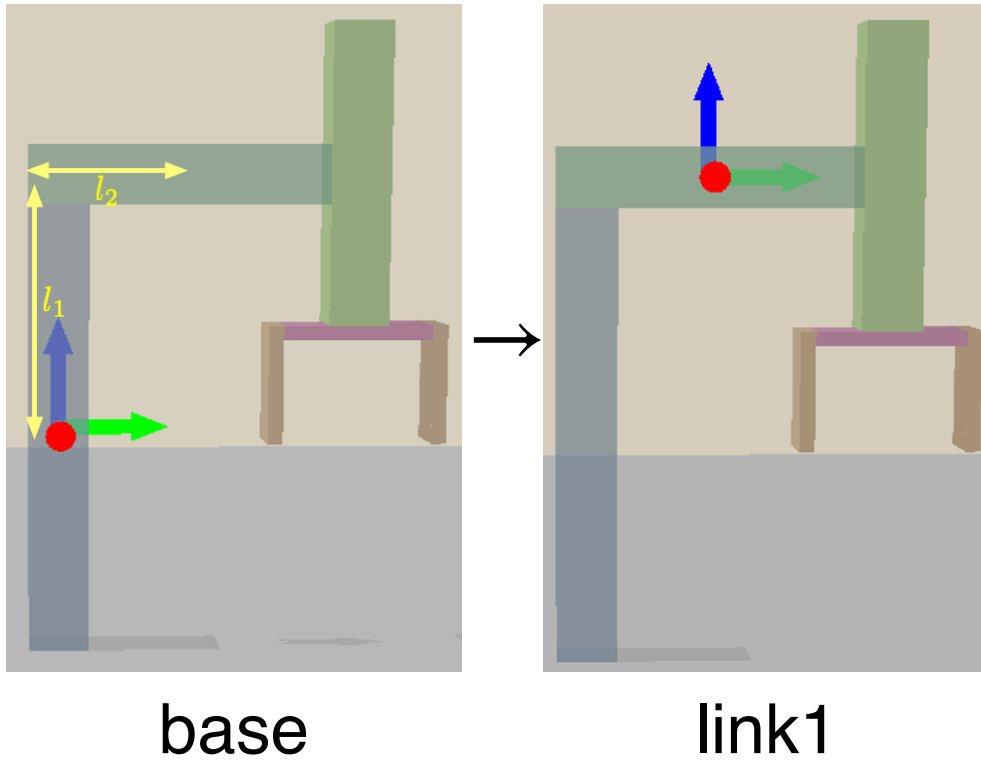
prismatic (θ_2)

Rotation Matrix

- Rotation about z axis is represented as:

$$R_z(\gamma) := \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

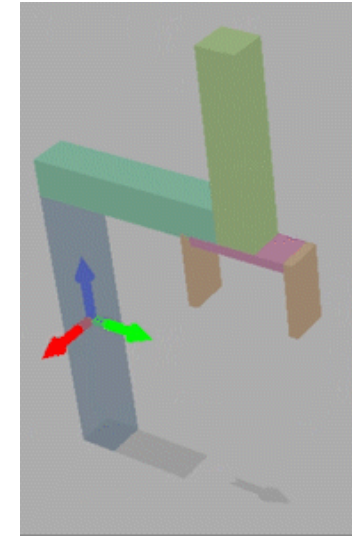
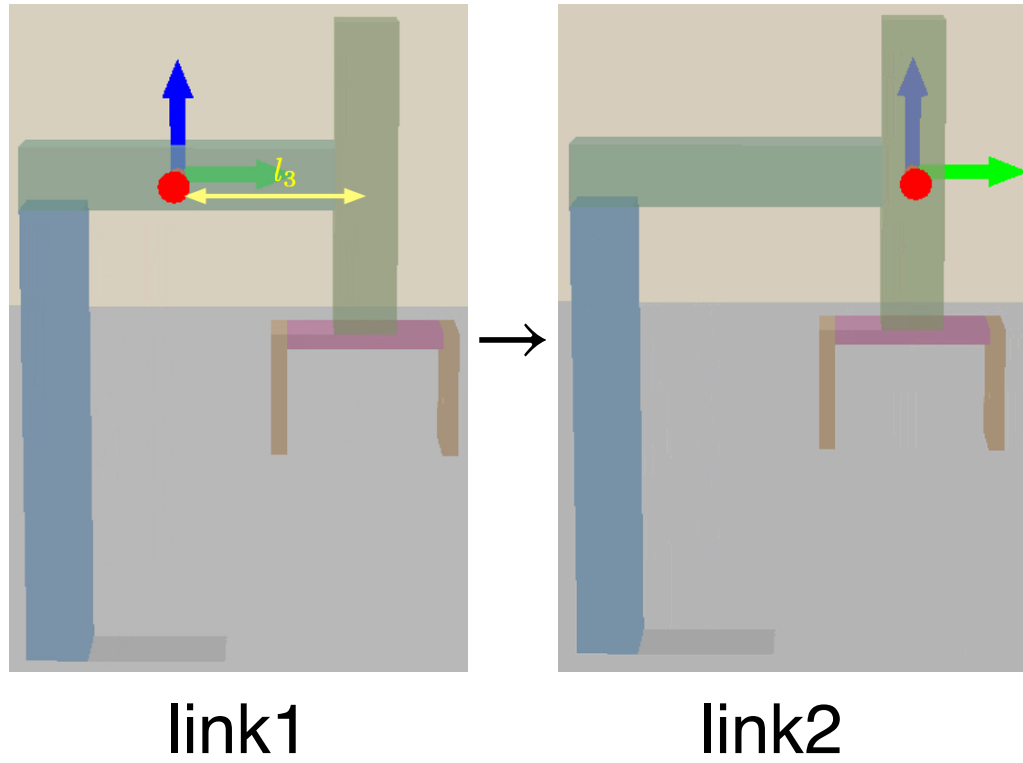
Examples



revolute (θ_1)

$$T_{0 \rightarrow 1}^0 = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 & -l_2 \sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 & 0 & l_2 \cos \theta_1 \\ 0 & 0 & 1 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

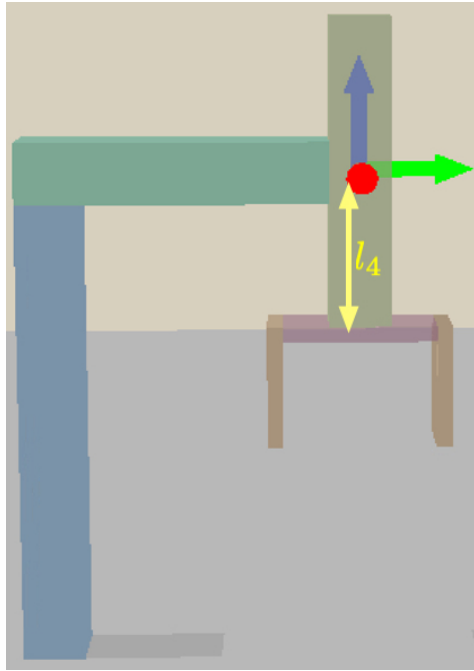
Examples



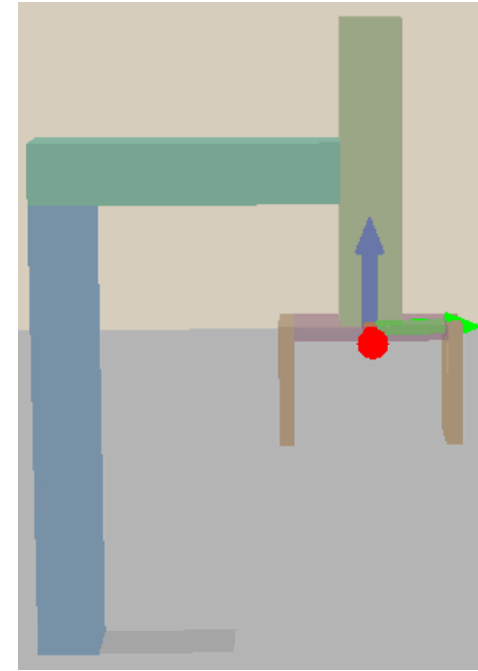
prismatic (θ_2)

$$T_{1 \rightarrow 2}^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_3 \\ 0 & 0 & 1 & \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Examples



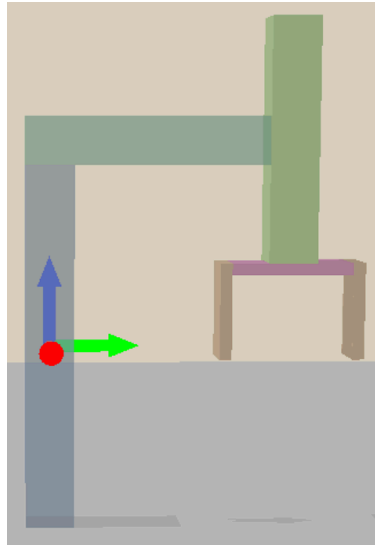
link2



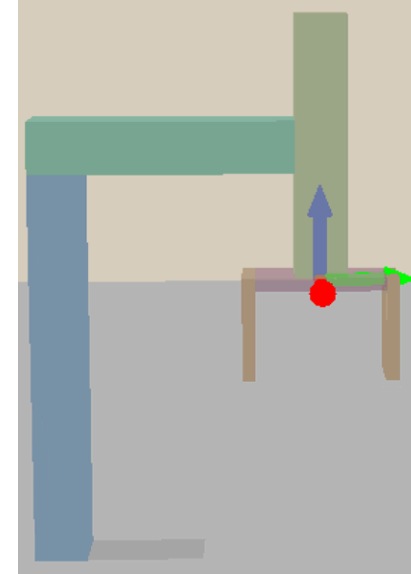
end_effector

$$T_{2 \rightarrow 3}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Kinematic Equations



base



end_effector

$$T_{0 \rightarrow 3}^0 = T_{0 \rightarrow 1}^0 T_{1 \rightarrow 2}^1 T_{2 \rightarrow 3}^2 = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 & -\sin \theta_1(l_2 + l_3) \\ \sin \theta_1 & \cos \theta_1 & 0 & \cos \theta_1(l_2 + l_3) \\ 0 & 0 & 1 & l_1 - l_4 + \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{s \rightarrow e}^s & \mathbf{t}_{s \rightarrow e}^s \\ 0 & 1 \end{bmatrix}$$

Kinematics Configuration

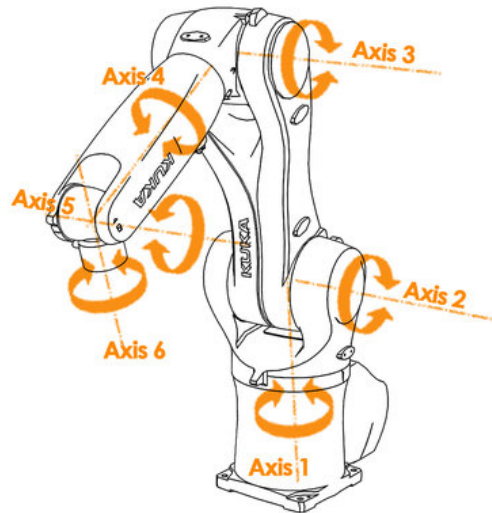
- Assuming frames are assigned to each link, we can parameterize **the pose of each joint**
 - Using the relative **angle** and **translation** between adjacent frames
- Two representations of the pose of the end-effector
 - **Joint space:** The space in which each coordinate is a vector of joint poses (**angles** around **joint axis**)
 - **Cartesian space:** The space of the rigid transformations of the end-effector by $(R_{s \rightarrow e}, \mathbf{t}_{s \rightarrow e})$, where $\mathcal{F}_e$ is the end-effector frame

Forward Kinematics

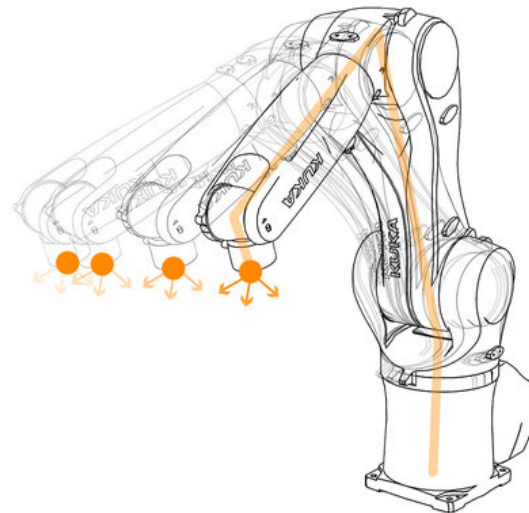
- Map the joint space coordinate $\theta \in \mathbb{R}^n$ to a transformation matrix T :

$$T_{s \rightarrow e} = f(\theta)$$

- Calculated by composing transformations along the kinematic chain



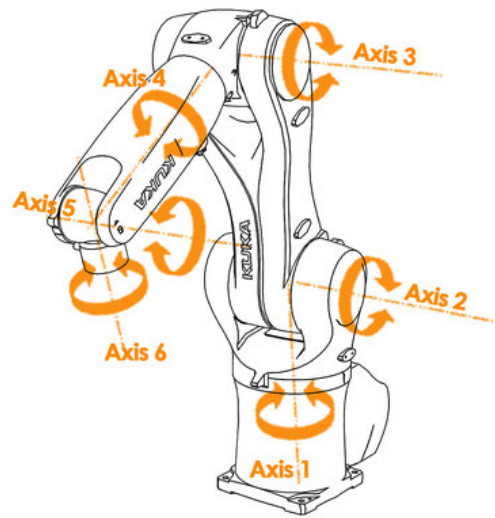
Forward Kinematic (FK) Solution



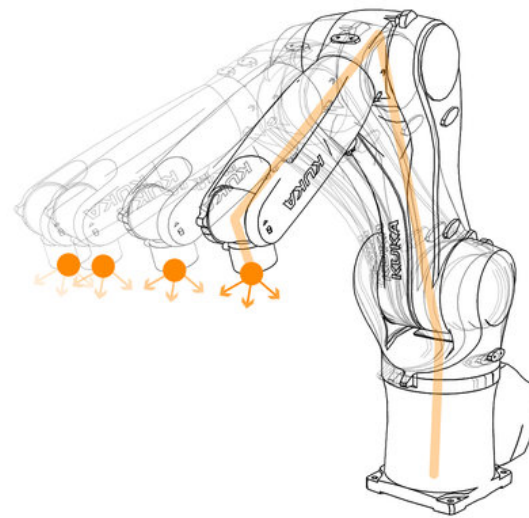
Inverse Kinematics (IK) Solution

Inverse Kinematics

- Given the forward kinematics $T_{s \rightarrow e}(\theta)$ and the target pose $T_{target} = \mathbb{SE}(3)$, find solutions θ that satisfy $T_{s \rightarrow e}(\theta) = T_{target}$



Forward Kinematic (FK) Solution



Inverse Kinematics (IK) Solution

IK Issues

- IK is challenging because while $T_{s \rightarrow e}(\theta)$ may be relatively easy to evaluate, T^{-1} usually isn't
- For T_{target} , there may be several possible solutions for T^{-1} , or there may be no solutions
- Even if there is a solution, it may require complex and expensive computations to find it
- As a result, there are many different approaches to solving IK problems

Why 6 DoF or 7 DoF arms?

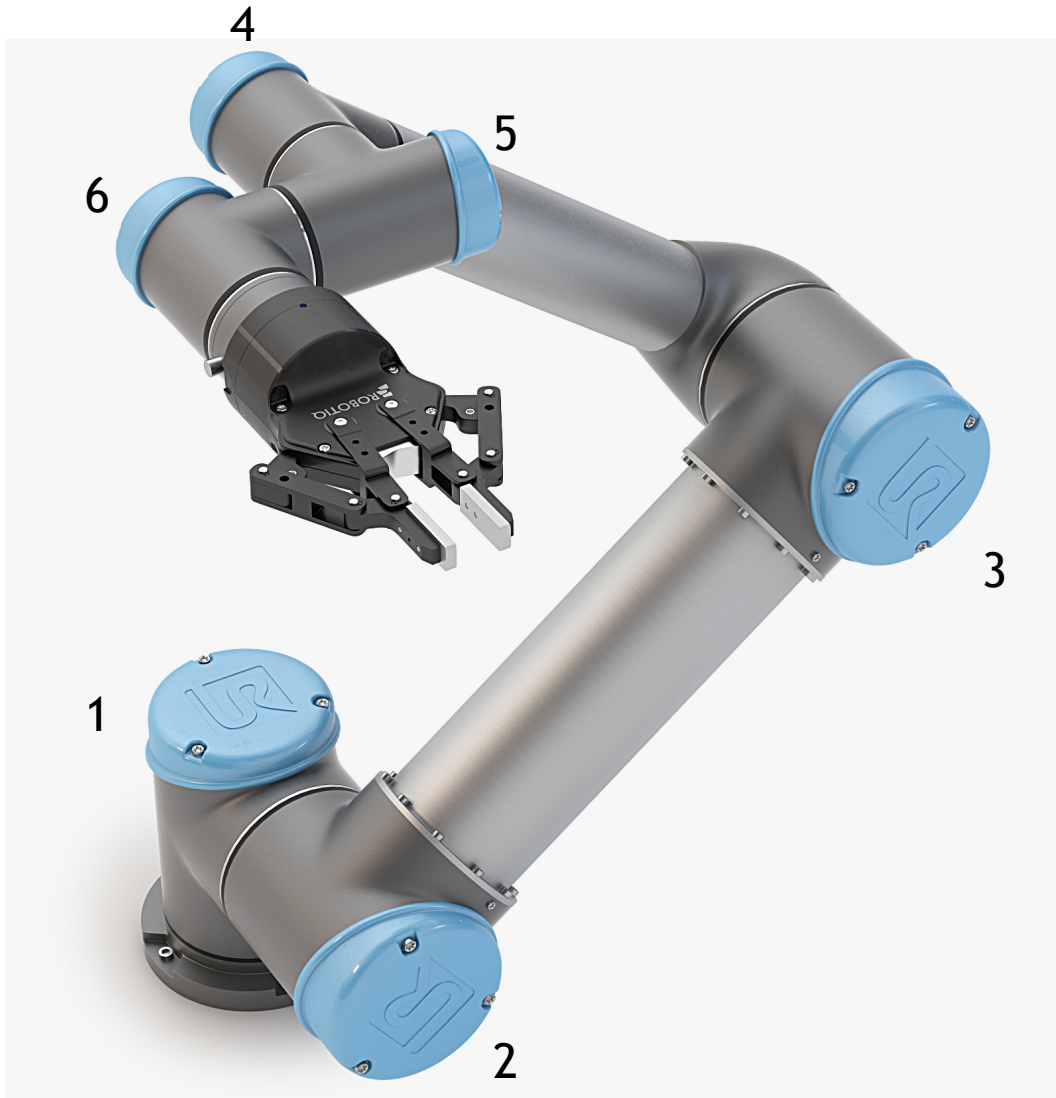
Analytical and Numerical Methods for IK

- One major way to classify IK solutions is into analytical and numerical methods
- Analytical methods attempt to mathematically solve an exact solution by directly inverting the forward kinematics equations. This is only possible on relatively simple chains.

Analytical Solution to IK

- The Pieper's criterion (proposed by Pieper.D.L in his 1968 paper "[THE KINEMATICS OF MANIPULATORS UNDER COMPUTER CONTROL](#)") is used to determine whether a closed-form inverse kinematics solution exists for a 6-DOF robotic arm.
- Specifically, when one of the following conditions is met, a 6-DOF kinematic structure has a closed-form inverse solution:
 - The axes of three consecutive rotational joints intersect at a single point.
 - The axes of three consecutive rotational joints are parallel.
- Note that this is only a sufficient condition but not necessary.

Example: UR5



- The axes of 2nd, 3rd, 4th joints are parallel.
- Thus meets Pieper's criterion.
- Closed-form solutions to IK exist.
- You can find the solutions in several μs .

Numerical Methods for IK

- Numerical methods use approximation and iteration to converge on a solution. They tend to be more expensive (usually take milliseconds), but far more general purpose.
- A numerical IK technique based on Jacobian matrices can be referred to https://cseweb.ucsd.edu/classes/wi17/cse169-a/slides/CSE169_08.pdf and https://cseweb.ucsd.edu/classes/wi17/cse169-a/slides/CSE169_09.pdf

Numerical Methods for IK

- Nvidia CuRobo and CuMotion packages implement collision-free IK solver. <https://youtu.be/Ux9tFBRLR-A>
- Solving with multiple different seeds (initial theta) in parallel on GPU.



How to Relate the Motion in Joint Space and Cartesian Space?

- Q1: If the robot moves by $\Delta\theta$ in the joint space, how much will it move in the Cartesian space? (Forward Kinematics)
- Q2: If the robot would move the end-effector by Δx in the Cartesian space, how shall it change its joint poses? (Inverse Kinematics)

$\mathrm{SO}(3)$ and $\mathrm{SE}(3)$

Rotation in $\mathbb{R}^3$



$\mathbb{SO}(3)$: The Space of Rotations

- $\mathbb{SO}(n) = \{R \in \mathbb{R}^{n \times n} : \det(R) = 1, RR^T = I\}$
- $\mathbb{SO}(n)$: “Special Orthogonal Group”
- “Group”: roughly, closed under matrix multiplication
- “Orthogonal”: $RR^T = I$
- “Special”: $\det(R) = 1$
- $\mathbb{SO}(2)$: 2D rotations, 1 DoF
- $\mathbb{SO}(3)$: 3D rotations, 3 DoF

$\text{SE}(3)$: The Space of Rigid Transformations

- $\text{SE}(3) := \left\{ T = \begin{bmatrix} R & \mathbf{t} \\ 0 & 1 \end{bmatrix}, R \in \text{SO}(3), \mathbf{t} \in \mathbb{R}^3 \right\}$
- $\text{SE}(3)$: “Special Euclidean Group”
 - “Group”: roughly, closed under matrix multiplication
 - “Euclidean”: R and $\mathbf{t}$
 - “Special”: $\det(R) = 1$
- 6 DoF

$\text{SO}(3)$ and $\text{SE}(3)$

- We need some theoretical understanding of $\text{SO}(3)$ and $\text{SE}(3)$
 - The parameterization

Parameterization

- Record $R \in \text{SO}(3)$ by real numbers
- In other words, we look for mappings between $\mathbb{R}^d$ and $\text{SO}(3)$:

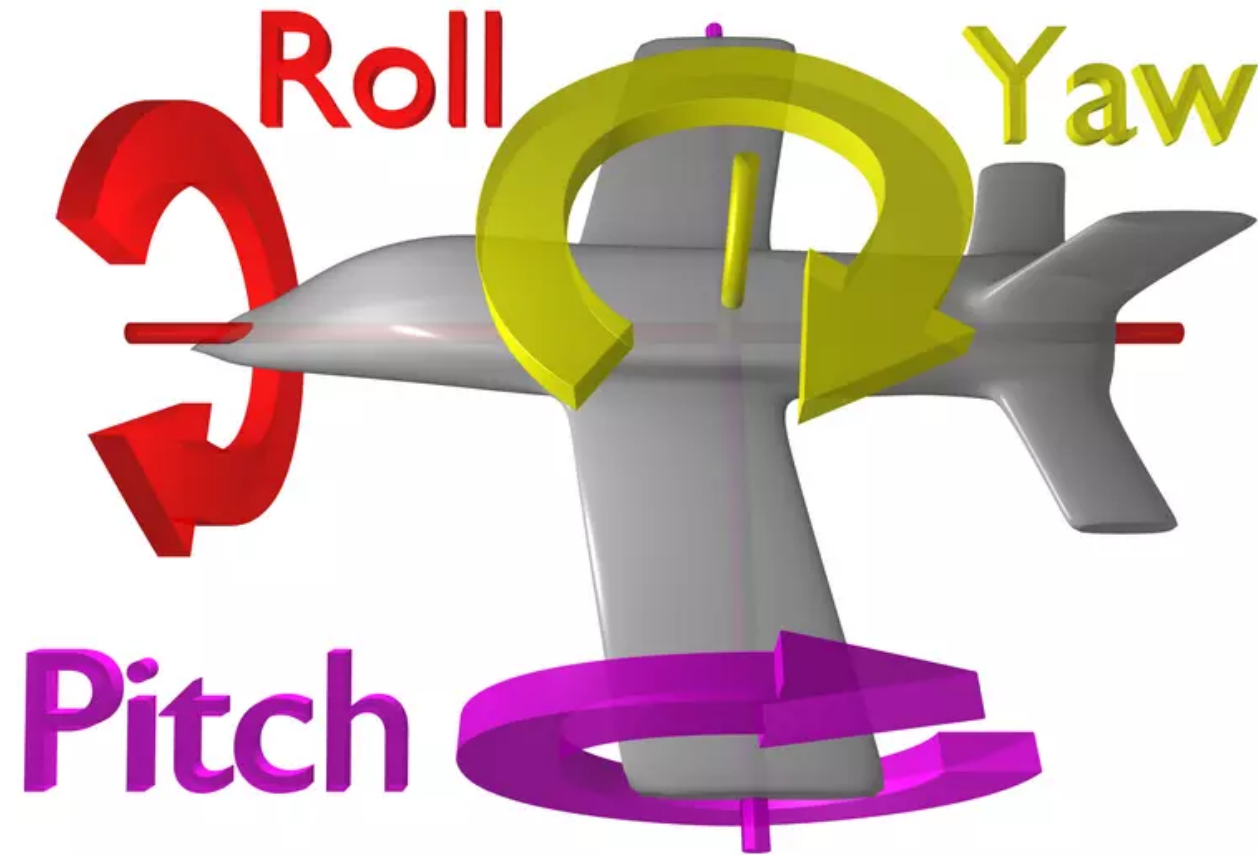
$$f(\theta) = R_\theta$$

Euler Angle

Euler Angle is Very Intuitive



Euler Angle



<https://blog.csdn.net/sandaiphon4869>

Euler Angle to Rotation Matrix

- Rotation about principal axis is represented as:

$$R_x(\alpha) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix}$$

$$R_y(\beta) := \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}$$

$$R_z(\gamma) := \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $R = R_z(\alpha)R_y(\beta)R_x(\gamma)$ for arbitrary rotation

Inspection

- Euler Angle is not unique for some rotations.
- For example,

$$R_z(45^\circ)R_y(90^\circ)R_x(45^\circ) = R_z(90^\circ)R_y(90^\circ)R_x(90^\circ)$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Gimbal Lock

- For example: When $\beta = \pi/2$,

$$R = R_z(\alpha)R_y(\pi/2)R_x(\gamma)$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ \sin(\alpha + \gamma) & \cos(\alpha + \gamma) & 0 \\ -\cos(\alpha + \gamma) & \sin(\alpha + \gamma) & 0 \end{bmatrix}$$

since changing α and γ has the same effects, a degree of freedom disappears!

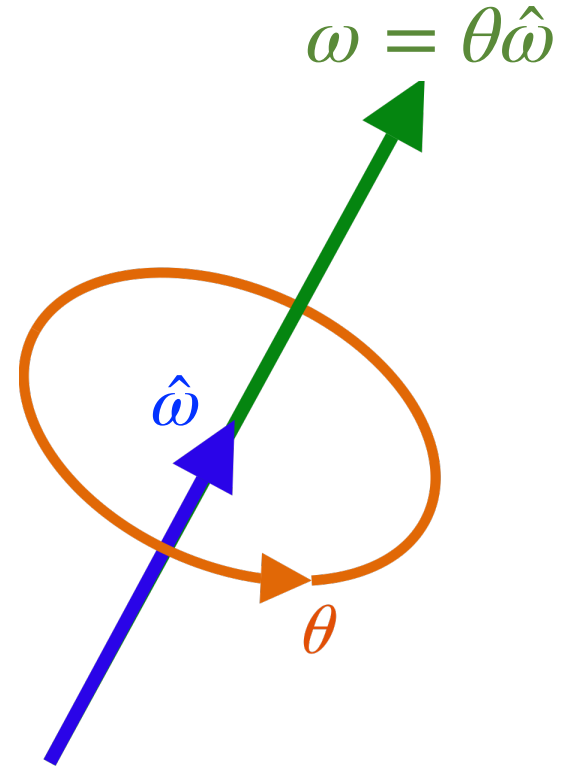
Summary

- Euler angle can parameterize every rotation and has good interpretability
- It is not a unique representation at some points
- There are some points where not every change in the target space (rotations) can be realized by a change in the source space (Euler angles)

Angle-Axis Parameterization of Rotations

Euler's Theorem

- Any rotation is equivalent to a rotation about a fixed axis $\hat{\omega} \in \mathbb{R}^3$ ($\|\hat{\omega}\| = 1$) through a positive angle θ
- $\hat{\omega}$: unit vector of rotation axis
- θ : angle of rotation
- $R \in \mathbb{SO}(3) := \text{Rot}(\hat{\omega}, \theta)$



Given $\hat{\omega}$ and θ , what is $R \in \mathbb{SO}(3)$?

Skew-Symmetric Matrix

- A is skew-symmetric $A = -A^T$
- Skew-symmetric matrix operator:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, [a] := \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

- Cross product can be a linear transformation
 - $a \times b = [a]b$

Given $\hat{\omega}$ and θ , what is $R \in \text{SO}(3)$?

- We can show that, for any $x \in \mathbb{R}^3$

$$\begin{aligned}\text{Rot}(\hat{\omega}, \theta)x &= x + (\sin \theta)\hat{\omega} \times x + (1 - \cos \theta)\hat{\omega} \times (\hat{\omega} \times x) \\ &= \{I + [\hat{\omega}]\sin \theta + [\hat{\omega}]^2(1 - \cos \theta)\}x\end{aligned}\quad (1)$$

Given $\hat{\omega}$ and θ , what is $R \in \text{SO}(3)$?

- We can show that, for any $x \in \mathbb{R}^3$
$$\begin{aligned}\text{Rot}(\hat{\omega}, \theta)x &= x + (\sin \theta)\hat{\omega} \times x + (1 - \cos \theta)\hat{\omega} \times (\hat{\omega} \times x) \\ &= \{I + [\hat{\omega}]\sin \theta + [\hat{\omega}]^2(1 - \cos \theta)\}x\end{aligned}\quad (1)$$
- By Taylor's expansion of **sin**, **cos**, $[\hat{\omega}]^3 = -[\hat{\omega}]$, and above

$$\text{Rot}(\hat{\omega}, \theta)x = (I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 + \frac{\theta^3}{3!}[\hat{\omega}]^3 + \dots)x$$

Given $\hat{\omega}$ and θ , what is $R \in \text{SO}(3)$?

- We can show that, for any $x \in \mathbb{R}^3$
$$\begin{aligned}\text{Rot}(\hat{\omega}, \theta)x &= x + (\sin \theta)\hat{\omega} \times x + (1 - \cos \theta)\hat{\omega} \times (\hat{\omega} \times x) \\ &= \{I + [\hat{\omega}]\sin \theta + [\hat{\omega}]^2(1 - \cos \theta)\}x\end{aligned}\quad (1)$$
- By Taylor's expansion of **sin**, **cos**, $[\hat{\omega}]^3 = -[\hat{\omega}]$, and above

$$\text{Rot}(\hat{\omega}, \theta)x = (I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 + \frac{\theta^3}{3!}[\hat{\omega}]^3 + \dots)x$$

- Recall Taylor's expansion of exponential,

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$

Given $\hat{\omega}$ and θ , what is $R \in \text{SO}(3)$?

- We can show that, for any $x \in \mathbb{R}^3$

$$\begin{aligned}\text{Rot}(\hat{\omega}, \theta)x &= x + (\sin \theta)\hat{\omega} \times x + (1 - \cos \theta)\hat{\omega} \times (\hat{\omega} \times x) \\ &= \{I + [\hat{\omega}]\sin \theta + [\hat{\omega}]^2(1 - \cos \theta)\}x \quad (1)\end{aligned}$$

- By Taylor's expansion of **sin**, **cos**, $[\hat{\omega}]^3 = -[\hat{\omega}]$, and above

$$\text{Rot}(\hat{\omega}, \theta)x = (I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 + \frac{\theta^3}{3!}[\hat{\omega}]^3 + \dots)x$$

- Recall Taylor's expansion of exponential,

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$

- Formally, we have:

$$\text{Rot}(\hat{\omega}, \theta)x = e^{[\hat{\omega}]\theta}x, \forall x \in \mathbb{R}^3$$

Given $\hat{\omega}$ and θ , what is $R \in \text{SO}(3)$?

- By $\text{Rot}(\hat{\omega}, \theta)x = e^{[\hat{\omega}]\theta}x, \forall x \in \mathbb{R}^3$,

$$\text{Rot}(\hat{\omega}, \theta) \equiv e^{[\hat{\omega}]\theta}$$

- This is under such a **Definition of Matrix Exponential**:

$$e^{[\hat{\omega}]\theta} = I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 + \frac{\theta^3}{3!}[\hat{\omega}]^3 + \dots$$

- By (1) in the last slide, we obtain **Rodrigues** formula:

$$e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\sin \theta + [\hat{\omega}]^2(1 - \cos \theta)$$

Given $\hat{\omega}$ and θ , what is $R \in \text{SO}(3)$?

- In the angle-axis representation of $\text{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta}$
- $\vec{\theta} = \hat{\omega}\theta$ is also called the **rotation vector**, or **exponential coordinate**

Rodrigues Formula

- Definition of Matrix Exponential:

$$e^{[\hat{\omega}]\theta} = I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 + \frac{\theta^3}{3!}[\hat{\omega}]^3 + \dots$$

- Sum of infinite series? **Rodrigues Formula**

- Can prove that $[\hat{\omega}]^3 = -[\hat{\omega}]$
- Then, use Taylor expansion of **sin** and **cos**
- $e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\sin \theta + [\hat{\omega}]^2(1 - \cos \theta)$

Given $\hat{\omega}$ and θ , what is $R \in \text{SO}(3)$?

- First question: Is there a **unique** parametrization?

- No:

1. $(\hat{\omega}, \theta)$ and $(-\hat{\omega}, -\theta)$ give the same rotation

2. when $R = I$, $\theta = 0$ and $\hat{\omega}$ can be arbitrary

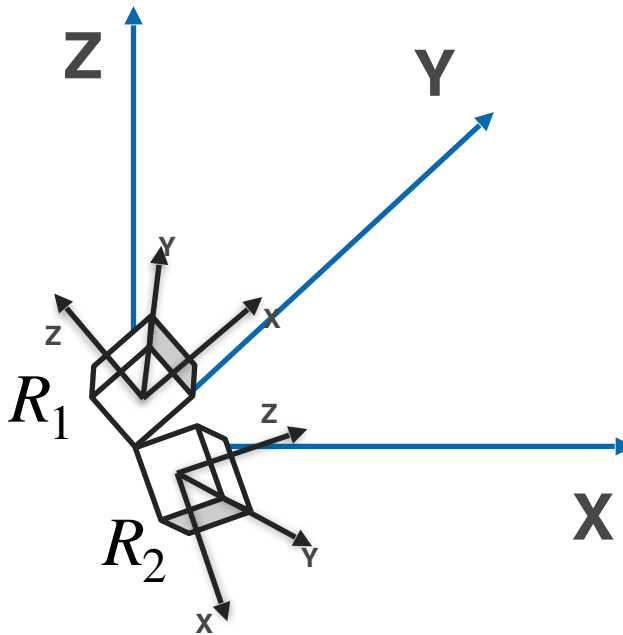
3. $(\hat{\omega}, \pi)$ and $(-\hat{\omega}, \pi)$ give the same rotation
($\text{tr}(R) = -1$)

- If we restrict $\theta \in (0, \pi)$, a unique parameterization exists:

- $\theta = \arccos \frac{1}{2}[\text{tr}(R) - 1], \quad [\hat{\omega}] = \frac{1}{2 \sin \theta}(R - R^T)$

Distance between Rotations

- How to measure the distance between rotations (R_1, R_2) ?
- A natural view is to measure the (minimal) effort to rotate the body at R_1 pose to R_2 pose:
 $\because (R_2 R_1^T) R_1 = R_2 \quad \therefore \text{dist}(R_1, R_2) = \theta(R_2 R_1^T) = \arccos \frac{1}{2} [\text{tr}(R_2 R_1^T) - 1]$





Robot Vision and Learning

– From a Perspective of Embodied AI

Next week: Lecture 3, Robotics II

